

# Kontsevich Integral for Vassiliev Invariants from Chern-Simons Perturbation Theory in the Light-Cone Gauge

J. M. F. Labastida<sup>a,b</sup> and Esther Pérez<sup>b</sup>

<sup>a</sup> *Theory Division, CERN,  
CH-1211 Geneva 23, Switzerland.*

<sup>b</sup> *Departamento de Física de Partículas,  
Universidade de Santiago de Compostela,  
E-15706 Santiago de Compostela, Spain.*

## Abstract

We analyse the structure of the perturbative series expansion of Chern-Simons gauge theory in the light-cone gauge. After introducing a regularization prescription that entails the consideration of framed knots, we present the general form of the vacuum expectation value of a Wilson loop. The resulting expression turns out to give the same framing dependence as the one obtained using non-perturbative methods and perturbative methods in covariant gauges. It also contains the Kontsevich integral for Vassiliev invariants of framed knots.

# 1 Introduction

Chern-Simons gauge theory has provided a very useful tool to study different aspects of the theory of knot and link invariants. Since its formulation by Witten in 1988 [1] it has been studied from both perturbative and non-perturbative points of view. While non-perturbative methods [1, 2, 3, 4, 5, 6] have led to its connection to polynomial invariants as the Jones polynomial [7] and its generalizations [8, 9, 10], perturbative ones [11, 12, 13, 14] have provided representations of Vassiliev invariants.

One of the advantages of the Chern-Simons approach to Vassiliev invariants is that it provides different representations of them because the perturbative analysis of the theory can be carried out in different gauges. Most of the perturbative analysis has been performed in covariant gauges, mostly in the Landau gauge. The loop structure in this gauge have been extensively analysed in [11, 15, 16, 17]. Vacuum expectation values of Wilson loops have also been studied in a variety of papers [11, 12, 13, 14, 18, 19]. These analyses lead to a representation of Vassiliev invariants for knots and links, which is equivalent to the one proposed by Bott and Taubes in [20].

The perturbative Chern-Simons series expansion in non-covariant gauges has not been studied so extensively. The loop structure has been analysed in the light-cone gauge in [21, 22]. A first step into the study of vacuum expectation values of Wilson loops was presented in [23], where a second-order analysis was carried out in the axial gauge. No systematic study of the perturbative series expansion corresponding to Wilson loops have been carried out in the light-cone gauge. The main goal of this paper is to begin with this study. In the process we find that the light-cone gauge leads to the Kontsevich integral [24, 25] for Vassiliev invariants of framed knots.

Chern-Simons gauge theory was first studied in the light-cone gauge in [4]. That paper points out a close relation between the vacuum expectation value of a Wilson line and the Knizhnik-Zamolodchikov equations [26]. This fact is used to prove that, for  $SU(N)$  as gauge group, vacuum expectation values of Wilson loops are related to the HOMFLY polynomial [8] for links. From a perturbative point of view, vacuum expectation values of Wilson loops have been considered only in [27], where it is conjectured that the corresponding perturbative series expansion is related to Kontsevich integral [24] for Vassiliev invariants. In the present paper we prove this conjecture.

Vacuum expectation values of Wilson loops are ill-defined in the light-

cone gauge because of the presence of singularities. To have a well-defined perturbative expansion we introduce framed Wilson loops. The fact that the right object to be studied in Chern-Simons gauge theory is a framed Wilson loop has been known since the theory was first formulated in [1]. The reason to introduce a framing might result different in each approach, but in general it is related to the presence of singularities in the  $n$ -point correlation functions at coincident points. This will indeed be the case in the light-cone gauge. A version of Kontsevich integral for Vassiliev invariants adapted to the case of framed knots and links was presented in [25]. In this paper we will consider the case of framed knots but a similar construction holds for framed links. We show that the vacuum expectation value of a Wilson loop contains the Kontsevich integrals introduced in [25].

The paper is organized as follows. In sect. 2 we discuss the quantization of Chern-Simons gauge theory in the light-cone gauge. In sect. 3 we analyse the general features of the perturbative series expansion of Wilson loops. In sect. 4, after introducing a regularization that involves framed knots, we prove the finiteness of the perturbative series expansion. In addition, we extract the framing dependence and show that the perturbative series contains the Kontsevich integral for Vassiliev invariants of framed knots. Finally, in sect. 5 we state our conclusions and we discuss some open problems and future work.

## 2 Chern-Simons gauge theory in the light-cone gauge

In this section we introduce Chern-Simons gauge theory in the light-cone gauge from a perturbation theory point of view. Let us consider a semi-simple compact Lie group  $G$  and a connection  $A$ . The action of the Chern-Simons theory over a three-dimensional Minkowski space  $\mathbf{M}^3$  is defined by the integral:

$$S_{\text{CS}}(A) = \frac{k}{4\pi} \int_{\mathbf{M}^3} \text{Tr} \left( A \wedge dA + \frac{2}{3} A \wedge A \wedge A \right), \quad (2.1)$$

where  $\text{Tr}$  denotes the trace over the fundamental representation of  $G$ , and  $k$  is a real parameter. This action is invariant under gauge transformation

$$A_\mu \rightarrow h^{-1} A_\mu h + h^{-1} \partial_\mu h, \quad (2.2)$$

where  $h$  is a map from  $\mathbf{M}^3$  to  $G$ , which is connected to the identity map. For maps that are not connected to the identity, (2.1) transforms into itself plus a term of the form  $2\pi k n$ ,  $n$  being the winding number of the map. If  $k$  is an integer, the exponential of the action,  $\exp(iS_{\text{CS}})$ , which is what enters the functional integral, is invariant under both types of gauge transformations.

Relative to the action (2.1) and to the gauge transformation (2.2), we choose the following conventions. The gauge group generators,  $T^a$ ,  $a = 1 \dots \dim(G)$ , will be taken anti-Hermitian, satisfying

$$[T^a, T^b] = -f_{abc} T^c, \quad (2.3)$$

$f_{abc}$  being the structure constants of the group  $G$ . They are normalized so that

$$\text{Tr}(T^a T^b) = -\frac{1}{2} \delta^{ab}, \quad (2.4)$$

in the fundamental representation. A point in  $\mathbf{M}^3$  will be denoted by coordinates  $x^\mu$ , with  $\mu$  running from zero to two,  $\mu = 2$  being the time coordinate.

In order to define the perturbative series expansion associated to the action (2.1) we must make a gauge choice. This choice is defined by a gauge-fixing condition, which in our case will be the corresponding to the light-cone gauge:

$$n^\mu A_\mu = 0, \quad (2.5)$$

where  $n^\mu$  is a constant vector satisfying  $n^2 = 0$ . The corresponding gauge-fixing term to be added to the action (2.1) is:

$$S_{\text{gf}} = \int_{\mathbf{M}^3} d^3x \text{Tr}(dn^\mu A_\mu + bn^\mu D_\mu c), \quad (2.6)$$

where  $d$  is an auxiliary field, and  $c$  and  $b$  are ghost fields;  $D_\mu$  stands for the covariant derivative,  $D_\mu c = \partial_\mu c + [A_\mu, c]$ . In order to define the perturbative series expansion it is convenient to rescale the fields by  $A \rightarrow gA$ , where  $g = \sqrt{\frac{4\pi}{k}}$ . The quantum action,  $S = S_{\text{CS}} + S_{\text{gf}}$ , becomes:

$$S = -\frac{1}{2} \int_{\mathbf{M}^3} d^3x \epsilon^{\mu\nu\rho} \left( A_\mu^a \partial_\nu A_\rho^a - \frac{g}{3} f_{abc} A_\mu^a A_\nu^b A_\rho^c \right) - \frac{1}{2} \int_{\mathbf{M}^3} d^3x (d^a A_\mu^a n^\mu + b^a n^\mu D_\mu^{ab} c^b), \quad (2.7)$$

where:

$$D_\mu^{ab} = \partial_\mu \delta^{ab} - g f_{abc} A_\mu^c. \quad (2.8)$$

Following [4] we introduce light-cone coordinates:

$$x^+ = x^1 + x^2, \quad x^- = x^1 - x^2, \quad (2.9)$$

and light-cone components for the gauge connection:

$$A_+ = A_1 + A_2, \quad A_- = A_1 - A_2. \quad (2.10)$$

Choosing the vector  $n^\mu$  as  $(0, 1, -1)$  the gauge-fixing condition (2.5) implies  $A_- = 0$ . Notice that the gauge condition (2.5) does not fix the gauge completely. For example, in our particular choice of  $n^\mu$  we still have gauge invariance under gauge transformations (2.2), in which the gauge parameter  $h$  depends only on  $x^0$  and  $x^+$ . The quantum action (2.7) takes the following form:

$$S = \int_{\mathbf{M}^3} d^3x (A_+^a \partial_- A_0^a - A_0^a \partial_- A_+^a - b^a \partial_- c^a). \quad (2.11)$$

Actually, to consider this action as the quantum action of the theory constitutes a rather simplified version of the full story. The perturbative higher-loop analysis in axial-type gauges is a very delicate issue, which requires to take into consideration some specific prescription to regulate unphysical poles. Fortunately, this analysis has been done for the case of Chern-Simons gauge theory in [21]. In these works it is shown that the effect of higher-loop contributions is a shift of the parameter  $k$  entering the Chern-Simons action

(2.1) by the quantity  $c_v$ , which denotes the value of the quadratic Casimir in the adjoint representation of the gauge group. Though strictly speaking this has been proved at one loop, it is believed that, as in the case of covariant gauges, it holds at any order in higher loops. In this paper we will assume that higher-loop effects just account for the shift of the parameter  $k$  in (2.1), and we will work with (2.11) as the quantum action of the theory.

The quantum action (2.11) has three important properties. First, it does not have derivatives in the transverse direction, second, it is quadratic in the fields, and, third, the ghost fields are not coupled to the gauge fields. This last property implies that the ghost fields can be integrated out trivially. The second property implies that there are no interaction vertices and therefore that all the correlation functions are determined by the two-point correlation functions. Certainly, this property notably simplifies the structure of the perturbative series expansion.

The two-point correlation functions corresponding to the gauge fields entering (2.11) have been computed in [4]. Their result is more conveniently expressed after performing a Wick rotation to Euclidean space  $\mathbf{R} \times \mathbf{C}$ . A point in Euclidean space will be denoted as  $(t, z)$ , where  $z = x^1 + ix^2$ . After introducing  $A_z = A_1 + iA_2$  and  $A_{\bar{z}} = A_1 - iA_2$  one finds [4]:

$$\begin{aligned}\langle A_{\bar{z}}^a(x) A_m^b(x') \rangle &= 0, \\ \langle A_m^a(x) A_n^b(x') \rangle &= \delta^{ab} \epsilon_{mn} \frac{1}{2\pi i} \frac{\delta(t-t')}{z-z'},\end{aligned}\tag{2.12}$$

with  $m, n = \{0, z\}$ , and  $\epsilon_{mn}$  is antisymmetric with  $\epsilon_{0z} = 1$ .

### 3 Perturbative expansion of the Wilson loop

Wilson loops are gauge-invariant operators of Chern-Simons gauge theory labelled by a loop  $C$  embedded in  $\mathbf{M}^3$  and a representation  $R$  of the gauge group  $G$ . They are defined by the holonomy along the loop  $C$  of the gauge connection  $A$ :

$$\mathcal{W}_R(C, G) = \left[ P_R \exp g \oint A \right], \quad (3.1)$$

where  $P_R$  denotes that the integral is path-ordered and that  $A$  must be considered in the representation  $R$  of  $G$ . In Chern-Simons gauge theory one considers vacuum expectation values of products of Wilson-line operators:

$$\begin{aligned} & \langle \mathcal{W}_{R_1}(C_1, G) \mathcal{W}_{R_2}(C_2, G) \dots \mathcal{W}_{R_n}(C_n, G) \rangle \\ &= \frac{1}{Z_k} \int [DA] \mathcal{W}_{R_1}(C_1, G) \mathcal{W}_{R_2}(C_2, G) \dots \mathcal{W}_{R_n}(C_n, G) e^{iS_k(A)}, \end{aligned} \quad (3.2)$$

where  $Z_k$  is the partition function of the theory:

$$Z_k = \int [DA] e^{iS_{CS}(A)}. \quad (3.3)$$

As shown in [1] the quantity (3.2) is a link invariant associated to a coloured  $n$ -component link whose  $j$ -component  $C_j$  carries the representation  $R_j$  of the gauge group  $G$ . In this paper we will consider only the vacuum expectation value of a single loop  $\langle \mathcal{W}_R(C, G) \rangle$ . This quantity can be expressed as a perturbative series expansion in the coupling constant  $g$ , which is the result of evaluating all the corresponding Feynman diagrams. The structure of these diagrams is obtained using standard field theory methods. A typical diagram of order  $i$  consists of a solid thick circle representing the oriented path  $C$ , with  $i$  propagators attached to it in a certain order (see some examples in fig. 1). A term in the sum of the perturbative series may be regarded as constructed out of two Feynman rules: the one corresponding to the propagator (2.12):

$$D_{mn}^{ab}(x - x') = \delta^{ab} \epsilon_{mn} \frac{1}{2\pi i} \frac{\delta(t - t')}{z - z'}, \quad (3.4)$$

while all other components of  $D_{\mu\nu}$  vanish; and the one corresponding to the vertex between the end-point of a propagator and the oriented path  $C$ :

$$V_i^{j\mu a}(x) = g(T_{(R)}^a)_i^j \int dx^\mu. \quad (3.5)$$

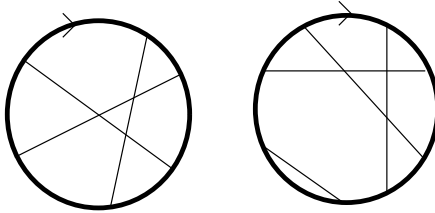


Figure 1: Examples of Feynman diagrams.

Notice that, as discussed in the previous section, there are not three-vertices in the light-cone gauge. The Feynman rules are depicted in fig. 2.

The power series expansion corresponding to  $\langle \mathcal{W}_R(C, G) \rangle$  can be written as [13]:

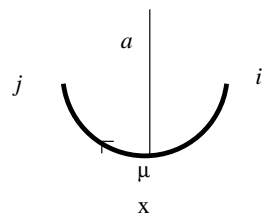
$$\langle \mathcal{W}_R(C, G) \rangle = \dim R \sum_{i=0}^{\infty} \sum_{j=1}^{d_i} \alpha_{ij}(C) r_{ij}(R) x^i, \quad (3.6)$$

where  $x = ig^2/2$  is the expansion parameter. The quantities  $\alpha_{ij}(C)$ , or geometrical factors, are combinations of path integrals of some kernels along the loop  $C$ , and the  $r_{ij}$  are traces of products of generators of the Lie algebra associated to the gauge group  $G$ . The index  $i$  corresponds to the order in perturbation theory, and  $j$  labels independent contributions at a given order,  $d_i$  being the number of these at order  $i$ . In (3.6)  $\dim R$  denotes the dimension of the representation  $R$ . Notice the convention  $\alpha_{01}(C) = 1$ . For a given order in perturbation theory,  $\{r_{ij}\}_{\{j=1\dots d_i\}}$  represents a basis of independent group factors.

The choice of basis in (3.6) is not unique. As shown in [19], there is a special type of basis, called canonical, for which the series (3.6) satisfies a factorization theorem. Let us introduce some notation to recall the definition of a canonical basis. A group factor is an element in the centre of the universal enveloping algebra of  $G$ , its general form being a trace over products of generators and structure constants with all indices contracted. It may also be represented in terms of diagrams that look like Feynman diagrams in which propagators and three-vertices are attached to a circle in a certain order, which represents the trace. In this way one considers three group-theoretical Feynman rules as the ones depicted in fig. 3. A canonical basis consists of an independent set of group factors made out of connected diagrams or products of non-overlapping connected subdiagrams (see [19] for more details).



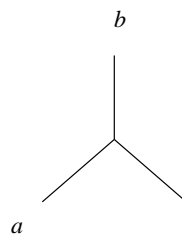
$$x \frac{a}{\mu} \frac{b}{\nu} x' = D_{\mu \nu}^{a b}(x - x')$$



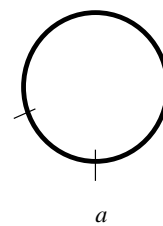
$$= V_i^{j \mu a}(x)$$

Figure 2: Feynman rules.

$$a \text{ --- } b = \delta^{ab}$$



$$= f_{abc}$$



$$= \text{Tr}(\mathbb{T}^a \dots)$$

Figure 3: Group-theoretical rules.

The factorization theorem [19] states that (3.6) may be written as an exponentiation over the elements of a canonical basis, which are not product of connected subdiagrams. These elements are called primitive and (3.6) takes the form:

$$\langle \mathcal{W}_R(C, G) \rangle = \dim R \exp \left\{ \sum_{i=0}^{\infty} \sum_{j=1}^{\hat{d}_i} \alpha_{ij}^c(C) r_{ij}^c(R) x^i \right\}. \quad (3.7)$$

In this expression the superindex  $c$  denotes the primitive connected diagrams. The quantities  $\alpha_{ij}^c(C)$  are their corresponding geometrical factors, which in general are linear combinations of the  $\alpha_{ij}(C)$  entering (3.6). In (3.7),  $\hat{d}_i$  denotes the number of independent primitive group factors at order  $i$ . For a more detailed discussion about the properties of (3.7) we refer the reader to [19]. Here we will recall only a few facts. In the series expansion (3.6) of  $\langle \mathcal{W}_R(C, G) \rangle$  there is a class of diagrams that contain at least one collapsible propagator. A propagator is called collapsible whenever its two legs are attached to two points in the oriented path  $C$ , which can be considered as the end-points of a part of  $C$  in which no other propagator is attached. Notice that in the right-hand side of (3.7) all these diagrams have factorized out as the exponentiation of the diagram of order 1, its geometrical factor being  $\alpha_{11}^c(C)$ . The perturbative analysis carried out in the Landau gauge in [13] demonstrated that if one chooses a canonical basis all the dependence on the framing is contained in  $\alpha_{11}^c(C)$ . In this paper we will show that, as expected, this also holds when the theory is analysed in the light-cone gauge.

As shown in [28, 29], each term of the perturbative series expansion (3.6) is a Vassiliev invariant. The quantities  $\alpha_{ij}^c(C)$ ,  $i > 1$ , in the exponential of (3.7) represent a particular choice of a basis of primitive ones. The analysis of the perturbative series expansion of the theory in the light-cone gauge will provide integral expressions for these invariants. Actually, in order to make contact with the Kontsevich integral we are not going to construct the integral expression for the primitive elements  $\alpha_{ij}^c(C)$ ,  $i > 1$ . We will not make the choice of a canonical basis except in our discussion on the framing dependence. Instead, we will analyse the perturbative series expansion as it results from the application of the Feynman rules without selecting a particular basis of group factors. As will be shown in the next section, this will lead us to Kontsevich integral for Vassiliev invariants.

Before entering into the analysis of the perturbative series expansion cor-

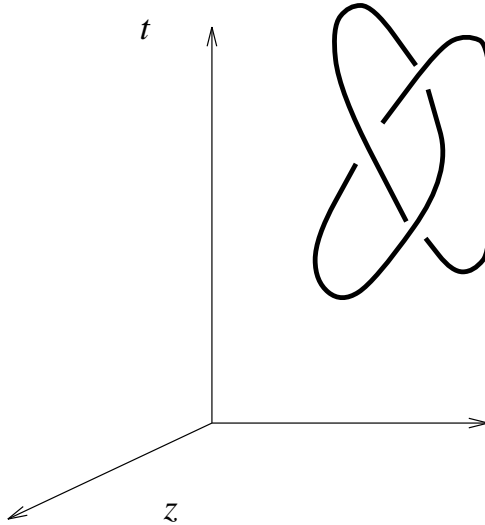


Figure 4: Example of a Morse knot.

responding to a Wilson loop, we must discuss the potential problems that might be encountered because of the particular form of the gauge-field propagator (3.4). This propagator is singular when its two end-points coincide. Actually, it is particularly singular in this situation, because both the numerator and the denominator lead to divergences. This fact tells us that, as in other gauges, one must consider Wilson loops with oriented paths  $C$  with no self-intersections. In other words, one must consider knots. However, contrary to covariant gauges, in the light-cone gauge we have two special kinds of singularities, there may be situations in which only one, the numerator or the denominator, leads to a divergence. In order to avoid singularities from the numerators one is forced to avoid paths with sections in which  $t$ , the first component of a generic point  $(t, z)$  in  $\mathbf{R} \times \mathbf{C}$ , is constant. This constraint, together with the fact that one is only allowed to consider the non-self-intersecting paths, implies that one must consider paths  $C$ , which correspond to Morse knots. A Morse knot in  $\mathbf{R} \times \mathbf{C}$  is a knot in which  $t$  is a Morse function on it. A Morse knot is characterized by  $2n$  extrema, half of them maxima, and the other half minima. An example of a Morse knot is depicted in fig. 4.

The third potential problem due to the structure of (3.4) comes from situations in which the two end-points of the propagator are close to one

of the extrema of a Morse knot since the denominator then vanishes. To solve this problem we will introduce a regularization procedure based on the introduction of a framing for the knot. The resulting invariants will correspond to invariants of framed knots.

## 4 The Kontsevich integral for framed knots

In this section we will apply the Feynman rules derived in the previous section to construct the perturbative series expansion corresponding to a Wilson loop for a Morse knot. As argued before, this requires the introduction of a regularization procedure; this will be achieved by considering framed knots. In the first part of this section we will describe this procedure in detail. In three subsequent subsections we will first prove the finiteness of the resulting perturbative series expansion, then we will extract the framing dependence, and, finally, we will obtain the Kontsevich integral for framed knots.

Let us consider a framed oriented Morse knot  $K$  on  $\mathbf{R} \times \mathbf{C}$ . The framing of  $K$  is defined by a vector  $m^\mu(K)$  normal to  $K$  such that along the coordinates  $(t(a), z(a))$ ,  $a \in S^1$ , which define  $K$ , we have another set of coordinates  $(t(a), z(a)) + \varepsilon(m^0(t, z), m^z(t, z))$ , which define a companion knot  $K_\varepsilon$ . For small  $\varepsilon$ ,  $K_\varepsilon$  is also a Morse knot, which does not intersect  $K$  and becomes  $K$  in the limit  $\varepsilon \rightarrow 0$ . We assign to  $K_\varepsilon$  the orientation that is naturally inherited from  $K$ . The framing is characterized by the vector  $m^\mu(K)$  or, equivalently, by the companion knot  $K_\varepsilon$ . It is natural to associate to each choice of framing the integer number that corresponds to the linking number,  $l$ , between  $K$  and  $K_\varepsilon$ .

Our regularization of the vacuum expectation value of a Wilson loop is based on the replacement of the propagator (3.4), which is attached to two points on  $K$ , by a propagator attached to a point in  $K$  and another in  $K_\varepsilon$ . As it is described below, there exists a precise way to carry out this replacement. Certainly, the resulting regularized integrals entering the power series expansion are finite for Morse knots. We have to show that they remain also finite in the limit  $\varepsilon \rightarrow 0$ . Before doing this, let us give full details of how the propagator replacement is carried out.

Let us assume that the Morse knot  $K$  possesses  $2n$  extrema. There are  $2n$  curves  $k^i$ ,  $i = 1, \dots, 2n$ , joining the different maxima and minima of  $K$ . For each curve  $k^i$  there is a one-to-one correspondence between points on  $k^i$  and the values that the variable  $t$  takes. The Morse knot can therefore be regarded as a complex multivalued function of the variable  $t$  with  $2n$  components, each one corresponding to a curve  $k^i$ , which is completely labelled by the values that the complex variable  $z$  takes in  $k^i$  as a function of  $t$ . One can think of two different parametrizations of  $K$ , the previous one:  $(t(a), z(a))$ ,  $a \in S^1$ , and a new one:  $z_i(t)$ ,  $t \in I^i$ ,  $i = 1, \dots, 2n$ , where  $I^i = [t_i^-, t_i^+]$  is the

segment of  $\mathbf{R}$  whose end-points are the values that the coordinate  $t$  takes at the two extrema joined by the curve  $k^i$ . A similar analysis can be done for the companion knot  $K_\varepsilon$ . It also possesses  $2n$  extrema and  $2n$  curves  $k_\varepsilon^i$ , parametrized by  $z'_i(t)$ ,  $t \in I_\varepsilon^i$ ,  $i = 1, \dots, 2n$ .

It is convenient to trade the integrations along the  $S^1$  parameter  $a$  of  $K$  by integrations over the height parameter  $t$ . Let us consider a propagator attached to the curves  $k^i$  and  $k^j$ . Using (3.4), it is easy to show that for such a propagator:

$$dx^\mu dx^\nu \langle A_\mu^a(x) A_\nu^b(x') \rangle \frac{1}{2\pi i} \rightarrow \frac{1}{2\pi i} dt_i dt_j \delta(t_i - t_j) p_{ij} \frac{\dot{z}_i(t_i) - \dot{z}_j(t_j)}{z_i(t_i) - z_j(t_j)}, \quad (4.1)$$

where:

$$p_{ij} = \begin{cases} 1 & \text{if } k_i \text{ and } k_j \text{ have the same orientation,} \\ -1 & \text{if } k_i \text{ and } k_j \text{ have opposite orientations.} \end{cases} \quad (4.2)$$

The propagators (4.1) may appear attached to two points lying on different curves  $k^i$  and  $k^j$ ,  $i \neq j$ , or to two points lying on the same curve  $k^i$ . In the second case the regularization consists of just replacing one of the points on  $k^i$  of the propagator by a point on  $k_\varepsilon^i$ . Since the delta function in the propagator (4.1) implies that its two end-points must be at the same height, there is no ambiguity in doing this. Notice, however, that in this case, since we have a path-ordered integration, when evaluating the delta function in (4.1) we must take into account the appearance of a factor  $1/2$ . In the first case,  $i \neq j$ , one uses the same procedure, namely one of the end-points is attached to the curve corresponding to the companion knot. However, there are now two possible ways of doing this. We will use the following prescription: we will add both possibilities and multiply the result by a factor  $1/2$ . For the same reasons as in the second case, this last step is well defined. Notice that in this case there is no additional factor  $1/2$  after the evaluation of the delta functions, since the variables of integration in each curve are now free.

The regularization prescription leads to the following formulae for the propagator (4.1). For  $i = j$ :

$$\frac{1}{2\pi i} \frac{1}{2} ds \frac{\dot{z}_i(s) - \dot{z}'_i(s)}{z_i(s) - z'_i(s)}, \quad (4.3)$$

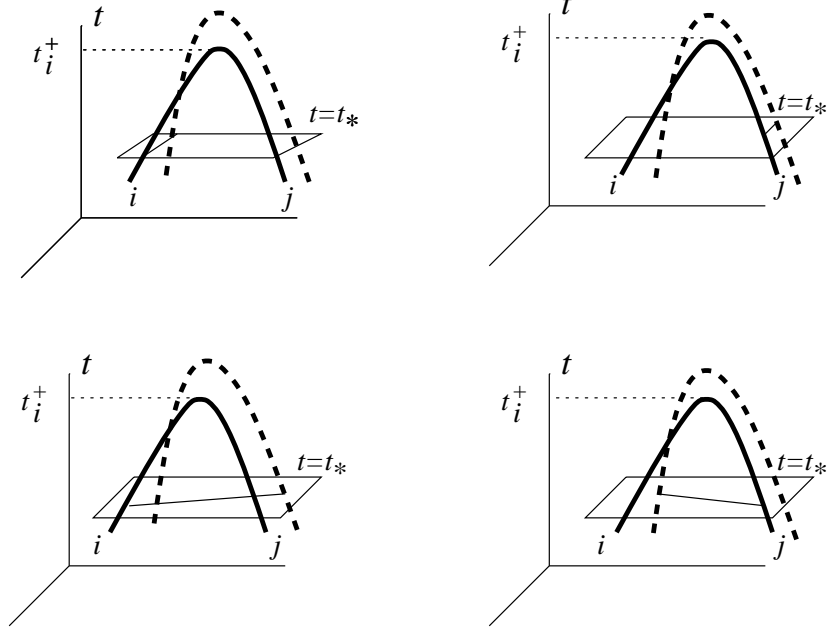


Figure 5: Generic divergent contributions.

while for  $i \neq j$ :

$$\frac{1}{2\pi i} \frac{1}{2} ds \left( \frac{\dot{z}_i(s) - \dot{z}'_j(s)}{z_i(s) - z'_j(s)} + \frac{\dot{z}'_i(s) - \dot{z}_j(s)}{z'_i(s) - z_j(s)} \right) p_{ij}. \quad (4.4)$$

## 4.1 Finiteness

We will now prove that our prescription is finite in the limit  $\varepsilon \rightarrow 0$ . There are four sources of singularities. All originate when the two end-points of a propagator are near an extremum. We will describe the situation for the case in which this extremum is a maximum, but it will become clear that a similar analysis can be carried out in the case in which it corresponds to a minimum. The four configurations that lead to divergences are depicted in fig. 5. We have drawn only the part of  $K$  around the maximum. We will assume that the rest of the diagram is the same in the four cases. Let us compute the four contributions after integrating from  $t_*$  to  $t_i^+$ . Using (4.3)

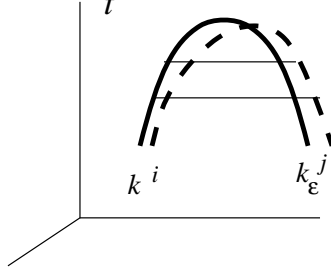


Figure 6: Higher-order divergent contributions.

and (4.4) one finds, after adding them up:

$$\frac{1}{2\pi i} \frac{1}{2} \log \frac{(z_i(s) - z'_i(s))(z_j(s) - z'_j(s))}{(z_i(s) - z'_j(s))(z'_i(s) - z_j(s))} \Big|_{s=t_*}^{s=t_i^+} \quad (4.5)$$

The most dangerous contribution is the one coming from the upper limit. However, since  $|z_i(t_i^+) - z'_i(t_i^+)| = |z_j(t_i^+) - z'_j(t_i^+)| = \varepsilon$ , and  $|z_i(t_i^+) - z'_j(t_i^+)| = |z_j(t_i^+) - z'_i(t_i^+)| = \varepsilon$ , one finds a cancellation of divergences in the limit  $\varepsilon \rightarrow 0$  and therefore the contribution is finite. One also encounters divergences from the lower limit of (4.5) in the  $\varepsilon \rightarrow 0$  limit. However, these divergences cancel against the ones originated after integration under the height  $t_*$ . One is therefore left with a term of the form:

$$-\frac{1}{2\pi i} \frac{1}{2} \log ((z_i(t_*) - z'_j(t_*))(z'_i(t_*) - z_j(t_*))), \quad (4.6)$$

which has to be taken into account when integrating over  $t_*$ . Notice that (4.6) is finite in the limit  $\varepsilon \rightarrow 0$  for values of  $t_*$  away from  $t_i^+$ , but it diverges when  $t_*$  approaches  $t_i^+$ . However, this divergence is too soft and it does not generate singularities in the integration, except in situations of the kind depicted in fig. 6. All the contributions of this type will be treated below when analysing the factorization of the framing dependence, and it will be shown that all possible divergences cancel out. We can therefore affirm that the perturbative series expansion corresponding to a Morse knot is finite in the limit  $\varepsilon \rightarrow 0$ .



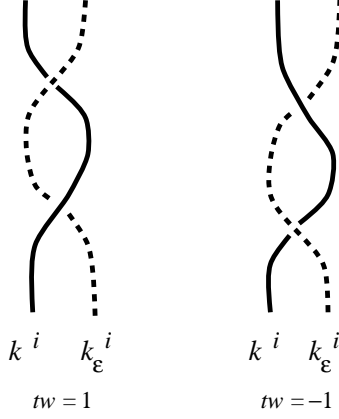


Figure 7: Basic contributions to the twist of a framed knot.

## 4.2 The framing contribution

In this subsection we are going to compute the lowest-order contribution to the perturbative series expansion of the vacuum expectation value of a Wilson loop. Using the factorization theorem discussed in sect. 3, we will obtain the full dependence on the framing. Our result can be stated very simply: if  $K$  is a framed Morse knot:

$$\langle \mathcal{W}_R(K, G) \rangle = e^{2\pi i l h} \langle \mathcal{W}_R(K, G) \rangle', \quad (4.7)$$

where  $h = \text{Tr}(T_{(R)}^a T_{(R)}^a)/k$ ,  $l$  is the linking number between  $K$  and its companion knot  $K_\varepsilon$ , and  $\langle \mathcal{W}_R(K, G) \rangle'$  is a framing-independent quantity. This result agrees with the one found non-perturbatively [1], and in covariant gauges [13].

The lowest-order contribution to the perturbative series expansion has the form:

$$ig^2 \frac{1}{2\pi i} \frac{1}{2} \text{Tr}(T_{(R)}^a T_{(R)}^a) \sum_{i,j=1}^{2n} \int_{\max\{t_i^-, t_j^-\}}^{\min\{t_i^+, t_j^+\}} ds \frac{\dot{z}_i(s) - \dot{z}_j'(s)}{z_i(s) - z_j'(s)} p_{ij}. \quad (4.8)$$

Notice that, owing to the prescription (4.4), the sum is over all possible values of  $i$  and  $j$  and not only for  $i \leq j$ . The factor  $i$  in front of this expression is due to the fact that one is considering the functional integral of  $i$  times the action (2.7). In general, in the perturbative series expansion, one must

include a factor  $i$  for each power of  $g^2$ . According to our previous arguments, the integral (4.8) is certainly finite in the limit  $\varepsilon \rightarrow 0$ . Actually it can be computed very easily. The key observation is that an integral of the form

$$\int_{t_*}^{t'_*} ds \frac{\dot{z}_i(s) - \dot{z}_j(s)}{z_i(s) - z_j(s)} \quad (4.9)$$

develops an imaginary part, which counts the number of times that the curve  $k_\varepsilon^j$  winds around the curve  $k^i$ , times  $2\pi$ . The real parts just cancel with each other, following a mechanism similar to the one described in our proof of finiteness. After summing over all the values of  $i$ , the contribution from the integral in (4.8) for  $i = j$  is just  $2\pi i$  times the twist  $tw$  of the band made by the two knots  $K$  and  $K_\varepsilon$ , *i.e.* the number of times that the band twists around itself. The basic contributions to  $tw$  are shown in fig. 7. The analysis of the contributions from  $i \neq j$  is very similar. The real parts cancel among themselves and some of the real contributions from the case  $i = j$ , as described in the proof of finiteness. The imaginary parts now count the number of times that the curve  $k^i$  twists around the curve  $k^j$ , times  $2\pi$ . Certainly, for this counting one can take the limit  $\varepsilon \rightarrow 0$ . If the Morse knot  $K$  is viewed in such a way that its projection in the plane corresponding to  $\text{Im } z = 0$  contains only single crossings, one can compute the contribution just assigning values  $\pm 1/2$  to the two types of crossings shown in fig. 8. The result of performing the integral in (4.8) is precisely  $2\pi i$  times the writhe  $w$  of the knot  $K$  for that given view. Notice that although we get factors of modulus  $1/2$  from each crossing, they are counted twice in (4.8). Thus, the full contribution from (4.8) is:

$$ig^2 \frac{1}{2} \text{Tr}(T_{(R)}^a T_{(R)}^a)(w + tw) = 2\pi i h l \quad (4.10)$$

since  $l = w + tw$ . In obtaining (4.10) we have used  $g^2 = 4\pi/k$ . Recall that  $h = \text{Tr}(T_{(R)}^a T_{(R)}^a)/k$ . Notice that, although  $w$  and  $tw$  depend on the view of the knot  $K$ , their combination  $l$ , the linking number between  $K$  and  $K_\varepsilon$ , is independent of it.

As shown in previous works [13, 19], the contribution found at first order exponentiates if one considers a canonical basis. Let us describe the reasons for this exponentiation by analysing what occurs at next order in perturbation theory. At second order one encounters the two group factors

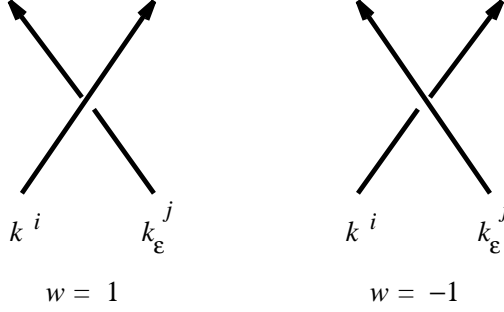


Figure 8: Basic contributions to the writhe.

corresponding to the Feynman diagrams shown in fig. 9. Diagrams *a* and *b* share the same group factor:

$$\text{Tr}(T_{(R)}^a T_{(R)}^a T_{(R)}^b T_{(R)}^b), \quad (4.11)$$

while diagram *c* possesses a different one:

$$\text{Tr}(T_{(R)}^a T_{(R)}^b T_{(R)}^a T_{(R)}^b). \quad (4.12)$$

If one decomposes this group factor in terms of the first one and a new group factor using the commutation relations (2.3):

$$\text{Tr}(T_{(R)}^a T_{(R)}^b T_{(R)}^a T_{(R)}^b) = \text{Tr}(T_{(R)}^a T_{(R)}^b T_{(R)}^b T_{(R)}^a) - f_{abc} \text{Tr}(T_{(R)}^a T_{(R)}^b T_{(R)}^c), \quad (4.13)$$

the full sum of the geometrical terms multiplying the first group factor can be written as  $1/2$  times a product of integrations over contributions coming from a single propagator. It precisely gives the second-order term of the exponential of (4.10). A similar rearrangement can be done at any order. Notice that in order to extract the terms building the exponential of (4.10) one modifies the group factors that remain in the rest. This mechanism is encoded in the factorization theorem [19]. Recall that in our analysis of finiteness we postponed the discussion on the cancellation of divergences from diagrams as the one in fig. 6. Now is the moment to discuss that issue. These diagrams are indeed the ones having group factors as (4.11). Together with the ones coming after rearranging group factors as done in (4.13), after adding them up, one obtains an expression that can be written as a product of the contribution from the first order. Since that first-order contribution is

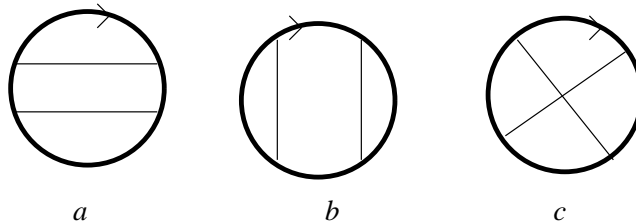


Figure 9: Feynman diagrams contributing at second order.

finite, the contribution at any arbitrary higher order is also finite. Notice also that this argument shows that all the dependence on the framing is contained in the exponential factor and therefore we can affirm that (4.7) holds. Indeed all the potential dependence on the framing must come from diagrams that lead to divergences. As we have seen, these diagrams only contribute to the exponential factor in (4.7). This behaviour is entirely similar to the one occurring in the Landau gauge [13].

### 4.3 The Kontsevich integral

In this subsection we will prove that the perturbative series expansion for the vacuum expectation value of the Wilson loop contains the Kontsevich integral for framed knots as presented in [25].

Let us begin by writing all the contributions to a given order  $m$ . To carry this out we must consider all possible ways of connecting  $2m$  points on the Morse knot by  $m$  propagators, following the regularization prescription described in the previous section, *i.e.* with one point of the propagator attached to  $K$  and the other to its companion knot  $K_\epsilon$ , and then path-order integrating. The path-order integration can be split into a sum such that in each term enters a path-ordered integration along  $2m$  curves among the set  $k^i, k_\epsilon^i$ ,  $i = 1, \dots, 2n$ . This set of curves builds the Morse knot under consideration. A given term in this sum might contain propagators joining  $k^i$  and  $k_\epsilon^i$ . In this case one must introduce a factor  $1/2$  as explained in our discussion leading to (4.3). The contributions coming from propagators joining  $k^i$  and  $k_\epsilon^j$ , with  $i \neq j$ , have also a factor  $1/2$  due to the double counting, as explained in the discussion of eq. (4.4). Accordingly, propagators joining different curves must be replaced by  $1/2$  the sum of their two possible choices of attaching their end-points.

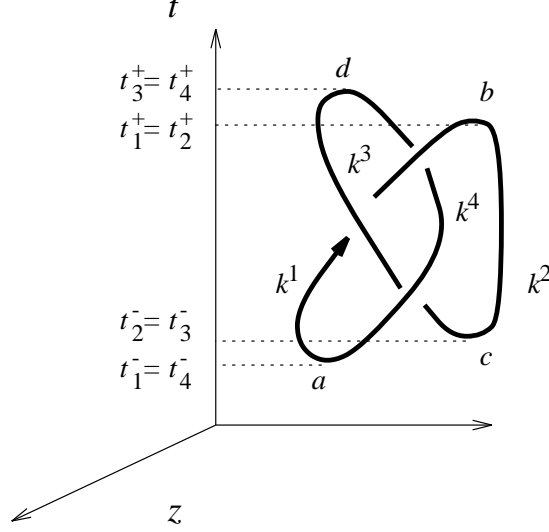


Figure 10: Labelled trefoil knot.

To each rearrangement of the  $m$  propagators corresponds a group factor. These are easily obtained using the group-theoretical Feynman rules shown in fig. 3. To fix ideas we will present in detail the second-order contribution,  $m = 2$ , for a particular group factor. For  $m = 2$  one must take into consideration the two group factors (4.11) and (4.12). As discussed above, the group factor (4.11) is associated to the framing dependence. We will analyse the contribution corresponding to the group factor (4.12). This contribution is of the form

$$(ig)^2 \int_0^1 ds_1 \int_{s_1}^1 ds_2 \int_{s_2}^1 ds_3 \int_{s_3}^1 ds_4 \dot{x}^{\mu_1}(s_1) \dot{x}^{\mu_2}(s_2) \dot{x}^{\mu_3}(s_3) \dot{x}^{\mu_4}(s_4) \\ \langle A_{\mu_1}^{a_1}(x(s_1)) A_{\mu_3}^{a_3}(x(s_3)) \rangle \langle A_{\mu_2}^{a_2}(x(s_2)) A_{\mu_4}^{a_4}(x(s_4)) \rangle \text{Tr}(T^{a_1} T^{a_2} T^{a_3} T^{a_4}). \quad (4.14)$$

We will now write more explicitly this multiple integral taking into account the form of the propagator (3.4). The delta function in this propagator imposes very strong restrictions on the possible contributions. Its presence implies that the only non-vanishing configurations are those in which the two end-points of each propagator are at the same height. To be more concrete let us consider the computation of (4.14) for the trefoil knot shown in fig. 10. This knot is made out of four curves,  $k^1$ ,  $k^2$ ,  $k^3$  and  $k^4$ , whose end-points are

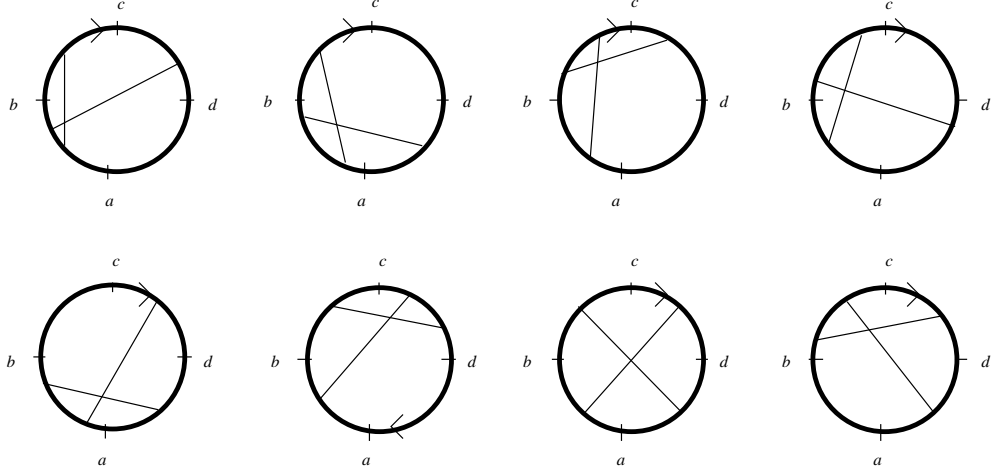


Figure 11: Contributions to a Morse knot made out of four curves.

the four critical points  $a$ ,  $b$ ,  $c$  and  $d$ . The heights of these critical points are:

$$\begin{aligned}
 a &\rightarrow t_1^- = t_4^-, \\
 b &\rightarrow t_1^+ = t_2^+, \\
 c &\rightarrow t_2^- = t_3^-, \\
 d &\rightarrow t_3^+ = t_4^+.
 \end{aligned} \tag{4.15}$$

They are depicted in fig. 10.

To obtain all the contributions we will divide in four parts the circle that represents the knot in fig. 10. Then we will join these parts by lines representing the propagators, taking into account the ordering of the four points to which they are attached. This ordering and the delta function in the height imply that no line can have its two end-points attached to the same part. They also imply that there are no contributions in which two end-points of different lines are attached to one part and the other two to another part. The only possibilities are shown in fig. 11. There is a total of eight contributions. Notice that this result is general for any Morse knot made out of four curves. The contributions are easily depicted on the knot itself as shown in fig. 12. For each contribution, one must compute a sign, which is the product of the  $p_{ij}$  in (4.2). The resulting signs are displayed in fig. 12. To be more explicit, let us write, for example, the integral associated

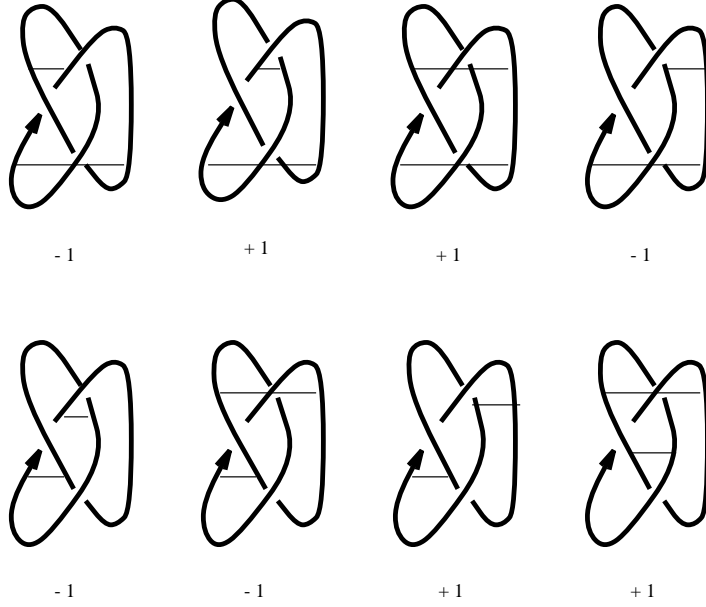


Figure 12: Contributions corresponding to the trefoil knot.

to the first contribution. It takes the form

$$(ig)^2 \frac{1}{(2\pi i)^2} \frac{1}{2^2} \int_{t_2^- < s_1 < s_2 < t_1^+} ds_1 ds_2 \left( \frac{\dot{z}_1(s_1) - \dot{z}_2'(s_1)}{z_1(s_1) - z_2'(s_1)} + \frac{\dot{z}_1'(s_1) - \dot{z}_2(s_1)}{z_1'(s_1) - z_2(s_1)} \right) \left( \frac{\dot{z}_3(s_2) - \dot{z}_1'(s_2)}{z_3(s_2) - z_1'(s_2)} + \frac{\dot{z}_3'(s_2) - \dot{z}_1(s_2)}{z_3'(s_2) - z_1(s_2)} \right). \quad (4.16)$$

Similar expressions can be easily written for the rest of the contributions depicted in fig. 12. The data entering in the double integral (4.16) are shown in fig. 13. Notice that this integral is not divergent if we take the limit  $\varepsilon \rightarrow 0$  before performing the integration. This feature is common to all the contributions corresponding to the group factor under consideration. As shown in our discussion on finiteness, only the contributions related to framing are potentially divergent. One could therefore remove in (4.16) the terms with primes and the factor  $1/2^2$ . The integral to be computed takes the form:

$$(ig)^2 \frac{1}{(2\pi i)^2} \int_{t_2^- < s_1 < s_2 < t_1^+} ds_1 ds_2 \frac{\dot{z}_1(s_1) - \dot{z}_2(s_1)}{z_1(s_1) - z_2(s_1)} \frac{\dot{z}_3(s_2) - \dot{z}_1(s_2)}{z_3(s_2) - z_1(s_2)}. \quad (4.17)$$

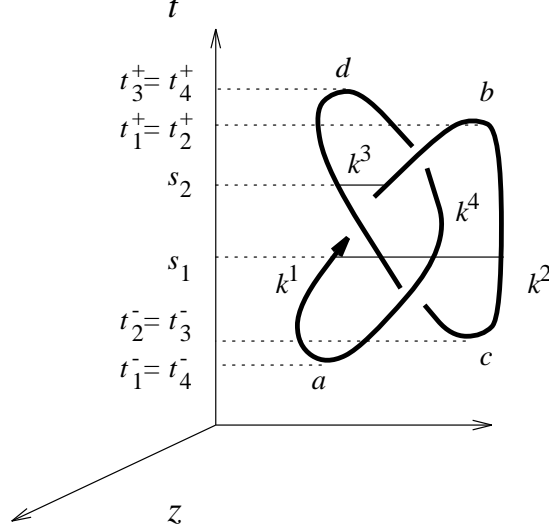


Figure 13: Details on the labeling of one of the contributions.

One of the two integrations can easily be performed, leading to:

$$(ig)^2 \frac{1}{(2\pi i)^2} \int_{t_2^- < s_2 < t_1^+} ds_2 \log \left( \frac{z_1(s_2) - z_2(s_2)}{z_1(t_2^-) - z_2(t_2^-)} \right) \frac{\dot{z}_3(s_2) - \dot{z}_1(s_2)}{z_3(s_2) - z_1(s_2)}. \quad (4.18)$$

Notice that, as argued before, this integral is finite. Although  $z_1$  and  $z_2$  get close to each other when  $s_2 \rightarrow t_1^+$ , the singularity in the integrand, being logarithmic, is too mild to lead to a divergent result. Expressions similar to (4.18) can easily be obtained for the rest of the contributions depicted in fig. 12.

We are now in a position to write the form of the general contribution originated from the Feynman rules of the theory. Notice that the most significant fact of our previous discussion is the presence of the delta function in the height of the propagator (3.4). It implies that the only non-vanishing configurations of the propagators are those in which their two end-points have the same height; in other words, only contributions in which the line representing the propagator is horizontal do not vanish. This observation allows us to rearrange the contributions to the perturbative series expansion in the following way. Consider all possible pairings  $\{z_i(s), z'_j(s)\}$  of curves  $k^i$  and  $k^j_\varepsilon$ ,  $i, j = 1, \dots, 2n$ , where  $2n$  is the number of extrema of the Morse



knot under consideration. A contribution at order  $m$  in perturbation theory will involve a path-ordered integral in the heights  $s_1 < \dots < s_r < \dots < s_m$  of a product of  $m$  propagators of the form (4.3) and (4.4):

$$\prod_{r=1}^m \frac{dz_{i_r}(s_r) - dz'_{j_r}(s_r)}{z_{i_r}(s_r) - z'_{j_r}(s_r)}. \quad (4.19)$$

This product is characterized by a set of  $m$  ordered pairings, each one labelled by a pair of numbers  $(i_r, j_r)$  with  $r = 1, \dots, m$ . We will denote an ordered pairing of  $m$  propagators generically by  $P_m$ . One must take into account all possible ordered pairings, *i.e.* one must sum over all the possible  $P_m$ . The group factor that corresponds to each ordered pairing  $P_m$  is simply obtained by placing the group generators at the end-points of the propagators and taking the trace of the product, which results after traveling along the knot. The resulting group factor will be denoted by  $R(P_m)$ .

Another ingredient in (4.4) that must be taken into account is the factor  $p_{ij}$ . For each pairing  $P_m = \{(i_r, j_r), r = 1, \dots, m\}$  there will be a contribution from their product. Certainly, the result will be a sign that will depend on the ordered pairing  $P_m$ . We will denote such a product by:

$$s(P_m) = \prod_{r=1}^m p_{i_r j_r}. \quad (4.20)$$

We are now in a position to write the full expression for the contribution to the perturbative series expansion at order  $m$ . It takes the form:

$$(ig^2)^m \left(\frac{1}{2\pi i}\right)^m \frac{1}{2^m} \sum_{P_m} \int_{t_{P_m}^- < t_1 < \dots < t_r < \dots < t_m < t_{P_m}^+} s(P_m) \prod_{r=1}^m \frac{dz_{i_r}(t_r) - dz'_{j_r}(t_r)}{z_{i_r}(t_r) - z'_{j_r}(t_r)} R(P_m), \quad (4.21)$$

where  $t_{P_m}^+$  and  $t_{P_m}^-$  are highest and lowest heights, which can be reached by the last and first propagators of a given ordered pairing  $P_m$ . This expression corresponds to the Kontsevich integral for framed knots as presented in [25].

## 5 Conclusions and open problems

In this paper we have presented an analysis of Chern-Simons gauge theory in the light-cone gauge from a perturbative point of view. As it became clear in our discussion of sect. 3, only vacuum expectation values associated to Morse knots are suitable for calculation in the light-cone gauge, at least if no additional regularization to the one considered in the paper is introduced. We have shown that the regularization prescription leads to the consideration of framed knots, and that the vacuum expectation value possesses a dependence on the framing that agrees with the ones appearing in other gauges and in the non-perturbative analysis of the theory. We have also shown that the perturbative series expansion contains the Kontsevich integral for framed knots.

Our results, however, demonstrate that something has been missed in the perturbative series expansion. Indeed, contrary to what is obtained in other gauges or in non-perturbative approaches, according to our result (4.21), the vacuum expectation value corresponding to the unknot shown in fig. 14 carrying a representation  $R$  of the gauge group is just  $\dim R$  (in the trivial framing). In other words, all the contributions in (4.21) vanish and one is left with the zeroth-order contribution, which is just  $\dim R$ . Actually, the expression (4.21) does not lead to knot invariants. It is well known [24] that the Kontsevich integral is only invariant under deformations of the knot which preserve the number of critical points. To obtain a truly invariant quantity one must take into account a correction to the sum of the contributions (4.21). If we denote by  $Z_m(K, K_\varepsilon)$  the contribution shown in (4.21) (order  $m$ ), the quantity that leads to a knot invariant is [24]:

$$\frac{\dim R + \sum_{m=1}^{\infty} Z_m(K, K_\varepsilon)}{\left(1 + \frac{1}{\dim R} \sum_{m=1}^{\infty} Z_m(U, U_\varepsilon)\right)^{\frac{n}{2}}} \quad (5.22)$$

where  $U$  and  $U_\varepsilon$  are the unknots shown in fig. 15, and  $n$  is the number of critical points of the knot  $K$ . The coefficients of the powers of  $g$  in this expression are Vassiliev invariants.

The fact that the perturbative series expansion corresponding to the terms (4.21) is not invariant under all kinds of deformations of the knot is something that one could have expected. As discussed in sect. 3, the perturbative construction fails for knots that are not of the Morse type. In

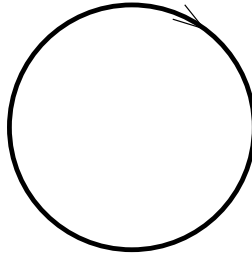


Figure 14: Unknot with two critical points.

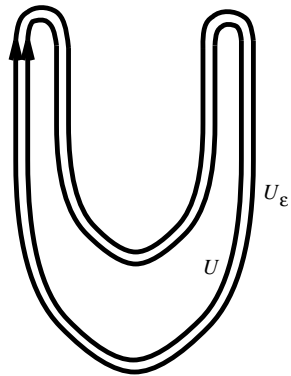


Figure 15: Unknots  $U$  and  $U_\varepsilon$  with four critical points.

a deformation of the knot in which the number of critical points is changed, one has to consider at some intermediate step a knot that is not of the Morse type. The analysis presented here is not applicable to that situation and it is therefore consistent to find that the final expression for the perturbative expansion is not invariant under those types of deformations. A very interesting problem is to understand, from the point of view of Chern-Simons gauge theory, why the denominator of (5.22) has to be included there. Presumably, this would reconcile the results obtained here with the ones from covariant gauges and from non-perturbative approaches. This and other related issues are at present under investigation.

### **Acknowledgements**

We would like to thank M. Alvarez for helpful discussions. This work was supported in part by DGICYT under grant PB93-0344.

## References

- [1] E. Witten, *Commun. Math. Phys.* **121** (1989) 351.
- [2] M. Bos and V.P. Nair, *Phys. Lett.* **B223** (1989) 61 and *Int. J. Mod. Phys.* **A5** (1990) 959; S. Elitzur, G. Moore, A. Schwimmer and N. Seiberg, *Nucl. Phys.* **B326** (1989) 108.
- [3] J.M.F. Labastida and A.V. Ramallo, *Phys. Lett.* **B227** (1989) 92 and **B228** (1989) 214; J.M.F. Labastida, P.M. Llatas and A.V. Ramallo, *Nucl. Phys.* **B348** (1991) 651; J.M.F. Labastida and M. Mariño, *Int. J. Mod. Phys.* **A10** (1995) 1045, hep-th/9402093; J.M.F. Labastida and E. Pérez, *J. Math. Phys.* **37** (1996) 2013, q-alg/9507031.
- [4] J. Frohlich and C. King, *Commun. Math. Phys.* **126** (1989) 167.
- [5] S. Martin, *Nucl. Phys.* **B338** (1990) 244.
- [6] R.K. Kaul, *Commun. Math. Phys.* **162** (1994) 289, hep-th/9305032; R.K. Kaul and T.R. Govindarajan, *Nucl. Phys.* **B380** (1992) 293, hep-th/9111063, and **B393** (1993) 392; P. Ramadevi, T.R. Govindarajan and R.K. Kaul, *Mod. Phys. Lett.* **A10** (1995) 1635, hep-th/9412084.
- [7] V. F. R. Jones, *Bull. AMS* **12** (1985) 103; *Ann. of Math.* **126** (1987) 335.
- [8] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K. Millett and A. Ocneanu, *Bull. AMS* **12** (1985) 239.
- [9] L.H. Kauffman, *Trans. Am. Math. Soc.* **318** (1990) 417.
- [10] Y. Akutsu and M. Wadati, *J. Phys. Soc. Jap.* **56** (1987) 839 and **56** (1987) 3039.
- [11] E. Guadagnini, M. Martellini and M. Mintchev, *Phys. Lett.* **B227** (1989) 111 and **B228** (1989) 489; *Nucl. Phys.* **B330** (1990) 575.
- [12] D. Bar-Natan, “Perturbative aspects of Chern-Simons topological quantum field theory”, Ph.D. Thesis, Princeton University, 1991.

- [13] M. Alvarez and J.M.F. Labastida, *Nucl. Phys.* **B395** (1993) 198, hep-th/9110069, and **B433** (1995) 555, hep-th/9407076; Erratum, *ibid.* **B441** (1995) 403.
- [14] D. Altschuler and L. Freidel, *Commun. Math. Phys.* **187** (1997) 261, q-alg/9603010.
- [15] L. Alvarez-Gaumé, J.M.F. Labastida and A. V. Ramallo, *Nucl. Phys.* **B334** (1990) 103; G. P. Korchemsky, *Mod. Phys. Lett.* **A6** (1991) 727.
- [16] A. Blasi, O. Piguet and S.P. Sorella, *Nucl. Phys.* **B356** (1991) 154.
- [17] C. P. Martin, *Phys. Lett.* **B241** (1990) 513; G. Giavarini, C.P. Martin, F. Ruiz Ruiz, *Nucl. Phys.* **B381** (1992) 222, hep-th/9206007.
- [18] M. Alvarez and J.M.F. Labastida, *Journal of Knot Theory and its Ramifications* **5** (1996) 779, q-alg/9506009; M. Alvarez, J.M.F. Labastida and E. Pérez, *Nucl. Phys.* **B488** (1997) 677, hep-th/9607030.
- [19] M. Alvarez and J.M.F. Labastida, “Primitive Vassiliev Invariants and Factorization in Chern-Simons Gauge Theory”, q-alg/9604010, to appear in *Commun. Math. Phys.*
- [20] R. Bott and C. Taubes, *J. Math. Phys.* **35** (1994) 5247.
- [21] G. Leibbrandt and C.P. Martin, *Nucl. Phys.* **B377** (1992) 593 and **B416** (1994) 351.
- [22] A. Brandhuber, S. Emery, M. Langer, O. Piguet, M. Schweda and S.P. Sorella, *Helv. Phys. Acta* **66** (1993) 551, hep-th/9305132.
- [23] J.F.W.H. van de Wetering, *Nucl. Phys.* **B379** (1992) 172.
- [24] M. Kontsevich, *Advances in Soviet Math.* **16**, Part 2 (1993) 137.
- [25] T.Q.T. Let and J. Murakami, “The universal Vassiliev-Kontsevich invariant for framed oriented links”, Max-Planck Institut für Mathematik preprint, 1994.
- [26] V.G. Knizhnik and A.B. Zamolodchikov, *Nucl. Phys.* **B247** (1984) 83.

- [27] A.S. Cattaneo, P. Cotta-Ramusino, J. Frohlich and M. Martellini, *J. Math. Phys.* **36** (1995) 6137, hep-th/9505027.
- [28] J.S. Birman and X.S. Lin, *Invent. Math.* **111** (1993) 225; J.S. Birman, *Bull. AMS* **28** (1993) 253.
- [29] D. Bar-Natan, *Topology* **34** (1995) 423.